Maggi's Equations of Motion and the Determination of Constraint Reactions

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This paper presents a geometrical derivation of the constraint reaction-free equations of Maggi for mechanical systems subject to linear (first-order) nonholonomic and/or holonomic constraints. These results follow directly from the proper application of the concepts of virtual displacement and quasicoordinates to the variational equation of motion, i.e., Lagrange's principle. The method also makes clear how to compute the constraint reactions (kinetostatics) without introducing Lagrangian multipliers.

I. Introduction

OVER the past few years, and in connection with the computational problems of multi-rigid-body dynamics, several papers dealing with the formulation of constraint *reaction-free* Lagrange-type equations of motion for systems subject to (linear) nonholonomic constraints have appeared (see, e.g., Refs. 1–4). The problem with these derivations, usually carried out without the explicit application of the concepts of virtual displacement and virtual work, is that they give the impression that their results are the fortuitous products of clever, but ad hoc algebraic manipulations on the Lagrange multiplier-containing equations of motion. What is missing there is not only geometrical insight, but primarily the relation of special results with the general principles of analytical mechanics (AM).

In the following we shall show how these general principles of analytical mechanics, when applied for the correct definitions of virtual displacement and virtual work and quasicoordinates, yield directly and naturally all kinds of reaction-free equations of motion. We shall also demonstrate how to use this "virtual" formalism to retrieve these constraint-caused reactions, if needed, without re-introducing Lagrangian multipliers, which constitutes another proof of the superiority of the geometrical and analytical approach. As an illustration, the reactions and equations of motion of the well-known knife edge or sled problem are determined.

II. Kinematical and Kinetical Background

Consider a general mechanical system whose configuration, relative to an inertial frame of reference F, is determined by a set of n generalized true or Lagrangian, independent, or unconstrained positional system coordinates

$$q = (q^1, \ldots, q^n) \tag{1}$$

all functions of time t. The position of a typical system particle P (relative to F) is

$$\boldsymbol{r} = \boldsymbol{r}\,(P;\, q,t) \equiv \boldsymbol{r}\,(q,t) \tag{2}$$

while its equation of motion (Newton's "second law")

$$\mathrm{d}m\boldsymbol{a} = \mathrm{d}\boldsymbol{f} + \mathrm{d}\boldsymbol{R} \tag{3}$$

where

$\mathrm{d}m$ = mass of P
$\boldsymbol{a}$ = acceleration of P (relative to F)
$\mathrm{d}\boldsymbol{f}$ = total impressed or applied (known) force on P
$\mathrm{d}\boldsymbol{R}$ = total reaction or constraint (unknown) force on P

The decomposition of the total force acting on P into $\mathrm{d}\boldsymbol{f}$ and $\mathrm{d}\boldsymbol{R}$, instead of the Newton-Euler decomposition of it into total external and total internal or mutual force, constitutes the contemporary interpretation of d'Alembert's initial idea. The absence of the explicit realization of this fundamental difference between the methods of Newton-Euler and d'Alembert-Lagrange is, and has been, the main cause of confusion in dynamics; this is not an academic point, but one loaded with practical consequences.

The system is subjected to m ($<n$) independent linear *nonholonomic* (= nonintegrable) kinematical constraints

$$\sum_{i=1}^{n} a_i^k \dot{q}^i + a^k = 0, \qquad (k = 1, \ldots, m) \tag{4}$$

the coefficients a_i^k, a^k are known or given functions of all q and t, in general, and taken together build an $m \times (n+1)$ matrix of rank m. Some, or all, of the constraints [Eq. (4)] may simply be holonomic ones in differential or kinematical form; most of what follows holds in this special case, too.

Now, the virtual displacement of a typical P, $\delta\boldsymbol{r}$, is defined as the linear or first-order part of

$$\Delta\boldsymbol{r} \equiv \boldsymbol{r}\,(q + \delta q, t) - \boldsymbol{r}\,(q,t)$$

where δq = differential increment of q compatible with the instantaneous or homogeneous, i.e., constant time, constraints. Thus, $\delta\boldsymbol{r}$ is the special differential

$$\delta\boldsymbol{r} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q^i}\,\delta q^i \equiv \sum_{i=1}^{n} \boldsymbol{e}_i \delta q^i \tag{5}$$

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where, however, the $n\,\delta q$ are not independent, but are restricted by the virtual form of the nonholonomic system constraints of Eq. (4):

$$\sum_{i=1}^{n} a_i^k \delta q^i = 0, \qquad (k=1,\ldots,m) \tag{6}$$

On the other hand, the particle and system "gradient" or holonomic (not necessarily unit) vectors $e_i \equiv \partial r/\partial q^i$ are linearly independent (since all nq are independent) and, therefore, for each particle P and at each "point" (q,t) of the system's (extended) "configuration space" V_{n+1}, build a (covariant) basis. The base vectors e_i are fundamental to all subsequent considerations. Considering formally (nonrelativistically!) the time as an additional $(n+1)$th system coordinate, i.e., setting always $q^{n+1} \equiv t$, one can extend the basis e to include $e_{n+1} \equiv \partial r/\partial q^{n+1} \equiv \partial r/\partial t$.

The basic postulate of analytical mechanics is "Lagrange's principle" (LP), or "d'Alembert's principle in Lagrange's form" (see Ref. 5, pp. 215–225 and 517–520). (Actually, LP is a *constitutive* postulate for the "geometrical" forces $\{-\mathrm{d}R\}$ and, as such, *not* a "law" of nature as, e.g., Newton's second law of motion!) According to LP, for any mechanical system subjected to ideal [or (virtual) work-less] and bilateral (reversible or two-sided) constraints

$$S(-\mathrm{d}R)\cdot\delta r = 0 \tag{7a}$$

or

$$S\,\mathrm{d}R\cdot\delta r = 0 \tag{7b}$$

where $S(\ldots)$ = summation over all of the particles of the discrete and/or continuous system, as in Stieltjes' integral. The $\{-\mathrm{d}R\}$ are called "lost" or "forlorn" forces. Thus, LP states that *the system of lost forces is in equilibrium,* not in the ordinary sense of zero force and torque, or even as in action-reaction, but in the virtual work sense of Eqs. (7). It should be stressed that it is the *sum* of the "lost virtual works" that vanishes and not necessarily the individual terms, although the latter may happen in special cases. Inserting $\mathrm{d}R$ from Eq. (3) into Eqs. (7) reduces it to the more familiar

$$S\,\mathrm{d}m a\cdot\delta r = S\,\mathrm{d}f\cdot\delta r \tag{8}$$

We shall call Eqs. (7) and (8) the "particle" or "vector" forms of LP. Next, substituting δr from Eq. (5) into Eqs. (7) and (8) yields, respectively,

$$\sum_{i=1}^{n} Q_i^{(r)}\delta q^i = 0 \tag{9}$$

$$\sum_{i=1}^{n} E_i(T)\delta q^i = \sum_{i=1}^{n} Q_i \delta q^i \tag{10}$$

where

$$S\,\mathrm{d}m a\cdot e_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \equiv E_i(T) \tag{11}$$

Lagrange's transformation [$E_i(\ldots)$ = Euler/Lagrange operator], and

$$T \equiv S\,\frac{1}{2}\,\mathrm{d}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2$$

which is equal to the kinetic energy (relative to F),

$$S\,\mathrm{d}f\cdot e_i \equiv Q_i \tag{12}$$

which is equal to the covariant ith component of impressed system force, and

$$S\,\mathrm{d}R\cdot e_i \equiv Q_i^{(r)} \tag{13}$$

which is equal to the covariant ith component of the system reaction force. The purely kinematical result [Eq. (11)] is standard fare to any advanced dynamics exposition, and is based on the kinematical equalities $e_i \equiv \partial r/\partial q^i = \partial v/\partial \dot{q}^i$, and $E_i(v) = (\partial v/\partial \dot{q}^i)^{\cdot} - \partial v/\partial q^i = 0$, where

$$v = \frac{\mathrm{d}r}{\mathrm{d}t} = \sum_{i=1}^{n} e_i\,\dot{q}^i + e_{n+1}$$

which is equal to the inertial velocity of P. The latter are valid for any constrained system, holonomic or not. (Similarly, $e_i = \partial a/\partial \ddot{q}^i$, where $a = \mathrm{d}v/\mathrm{d}t$ = acceleration. Clearly, this holds for any-order time derivative of r with respect to the same order time derivative of q^i.) Some references, e.g., Ref. 6, imply that Eq. (11) is valid only for holonomic systems. Also, Eq. (11) is a welcome but *secondary* result of Lagrangian dynamics and by no means its essence. The preceding results and definitions are valid for any ideal system, holonomic or not. Now, in nonholonomic systems, due to Eq. (6), the $n\,\delta q$ are not independent, and so one cannot deduce from Eqs. (9) and (10) that

$$Q_i^{(r)} = 0 \tag{14}$$

$$E_i(T) = Q_i, \qquad i=1,\ldots,n \tag{15}$$

These two sets of equations are valid only for independent δq. When constraints such as Eq. (4) or (6) are present, to make further advances from Eqs. (9) and (10) one proceeds in one of the following two ways: 1) Attach or adjoin Eq. (6) to Eqs. (9) and (10) via m Lagrangian multipliers $\lambda_k = \lambda_k(t)$, which makes $n\;\delta q$ independent again and, thus, yields the well-known Routh (1877)-Voss (1885) equations:

$$Q_i^{(r)} = \sum_{k=1}^{m} \lambda_k\, a_i^k \tag{16}$$

$$E_i(T) = Q_i + Q_i^{(r)}, \qquad (i=1,\ldots,n) \tag{17}$$

or 2) introduce new sets of $(n-m)$ independent virtual variations that automatically satisfy the constraints of Eq. (6); this is the method of quasicoordinates, or of constraint embedding (to the problem).

The ability to attach or adjoin the homogeneous constraints of Eq. (6) to the homogeneous form of Eqs. (9) and (10), i.e., the ability to use Lagrange's multipliers, is the key reason why one reads in the literature that the various δq must be small or infinitesimal. Usually this is repeated uncritically as an article of faith or as some dusty rule out of the past and, thus, adds to the considerable confusion regarding virtual displacements, e.g., Ref. 7, pp. 1076–1077 and Ref. 8. As long as the constraint equations have been brought to the linear and homogeneous (virtual) forms [Eq. (6)], and can, therefore, be adjoined to the variational equations of motions (9) and (10) via multipliers, the size of each δq does not matter; it is their ratios with each other that matters, as in ordinary differential calculus, i.e., the coefficients a_i^k. Thus for *linear* nonholonomic constraints there is no problem with the size of the δq. Problems do appear, however, in the case of *nonlinear* nonholonomic constraints such as

$$f_k(q,\,\dot{q},\,t) = 0, \qquad (k=1,\ldots,m) \tag{18}$$

and, unless one carefully defines virtual displacements, no further advance toward the formulation of reactionless system equations of motion can be made; i.e., Eqs. (18) cannot be attached in that form, or even as

$$\delta f_k = \sum_i \left(\frac{\partial f_k}{\partial q^i}\,\delta q^i + \frac{\partial f_k}{\partial \dot{q}_i}\,\delta \dot{q}^i\right)$$

to Eqs. (9) and (10). On this often neglected and misunderstood point see Ref. 9, p. 47 and Ref. 10, pp. 453-520, footnote on p. 454.

Following Hamel (Ref. 5, p. 473 et seq.) we introduce a set of n total derivatives of quasicoordinates θ or "quasivelocities" ω^k or, more precisely, contravariant nonholonomic components of system velocity at (q,t) in the extended configuration space V_{n+1} via the linear invertible transformations

$$\omega^k \equiv \sum_{i=1}^{n} a_i^k \dot{q}^i + a_{n+1}^k = 0, \qquad (k=1,\ldots,m)$$

$$\equiv \sum_{i=1}^{n} a_i^k \dot{q}^i + a_{n+1}^k \neq 0, \qquad (k=m+1,\ldots,n)$$

$$(\omega^{n+1} \equiv \dot{q}^{n+1} = \dot{t} = 1) \tag{19}$$

where now the nonsingular $(n+1)\times(n+1)$ matrix $\mathbf{a} = \{a_i^k, a_{n+1}^k$, plus the appropriately defined $a_k^{n+1}, a_{n+1}^{n+1}\}$ consists of the original $m\times(n+1)$ submatrix of Eq. (4) plus additional arbitrary elements (to make $\mathbf{a}$ square), but such that when the system (19) is solved for the $n\dot{q}$ as functions of the $(n-m)$ ω and the elements of $\mathbf{a}$ are substituted back into Eq. (4), they satisfy Eq. (4) identically. Carrying out this inversion yields

$$\dot{q}^i = \sum_{k=m+1}^{n} A_k^i \omega^k + A_{n+1}^i, \qquad (i=1,\ldots,n) \tag{20}$$

It can be readily shown that the following "compatibility" conditions hold

$$\sum_{i=1}^{n} a_i^k A_l^i = \delta_l^k, \qquad \sum_{i=1}^{n} a_k^i A_i^l = \delta_k^l, \qquad (k,l=1,\ldots,n) \tag{21a}$$

$$\sum_{k=1}^{n} a_{n+1}^k A_k^i = -A_{n+1}^i, \qquad \sum_{i=1}^{n} a_i^k A_{n+1}^i = -a_{n+1}^k,$$

$$(k=1,\ldots,n) \tag{21b}$$

where

$$a_k^{n+1} = \delta_k^{n+1}, \qquad A_k^{n+1} = \delta_k^{n+1}$$

and $k=1,\ldots,n;\ n+1;\ \delta_l^k$, etc., = the Kronecker delta. The virtual forms of Eqs. (19) and (20) are, respectively,

$$\delta\theta^k = \sum_{i=1}^{n} a_i^k \delta q^i = 0, \qquad (k=1,\ldots,m) \tag{22a}$$

$$\delta\theta^k = \sum_{i=1}^{n} a_i^k \delta q^i \neq 0, \qquad (k=m+1,\ldots,n) \tag{22b}$$

and

$$\delta q^i = \sum_{k=m+1}^{n} A_k^i \delta\theta^k, \qquad (i=1,\ldots,n) \tag{23a}$$

$$\delta q^{n+1} = \delta\theta^{n+1} = \delta t = 0, \qquad \text{(by LP)} \tag{23b}$$

where $\delta\theta$ = virtual variation or virtual differential of the quasicoordinate θ, and $\omega^k \equiv \mathrm{d}\theta^k/\mathrm{d}t \equiv \dot{\theta}^k$ (no ordinary derivative, since θ^k does not exist.)

Now we are ready to continue with the kinetic equations (7-10). Let us start with the particle ones. Inserting Eq. (23) into Eq. (5) yields

$$\delta \boldsymbol{r} = \sum_{i=1}^{n} \boldsymbol{e}_i \, \delta q^i$$

$$= \sum_{i=1}^{n} \boldsymbol{e}_i \left(\sum_{k=m+1}^{n} A_k^i \delta\theta^k \right)$$

$$= \sum_{k=m+1}^{n} \boldsymbol{a}_k \, \delta\theta^k \tag{24}$$

where

$$\boldsymbol{a}_k \equiv \sum_{i=1}^{n} A_k^i \, \boldsymbol{e}_i \tag{25}$$

The transformation equation (25) holds, of course, for $k=1,\ldots,m$, $m+1,\ldots,n$ (and even $n+1$). The representation of Eq. (24) is fundamental to AM.

The (covariant) particle *and* system basis vectors $\boldsymbol{a}_k$ are the nonholonomic counterparts of $\boldsymbol{e}_i$. Since the various $\boldsymbol{e}$ are linearly independent and the $\boldsymbol{e} \to \boldsymbol{a}$ transformation is nonsingular, the various $\boldsymbol{a}$ are also linearly independent, but contrary to the gradient $\boldsymbol{e}_i$, $\boldsymbol{a}_k$ (at least one of them) are nongradient; i.e., $\boldsymbol{a}_k \neq \partial \boldsymbol{r}/\partial$ (true coordinate)^k. One can, however, define the $\boldsymbol{a}_k$ by the symbolic operation, or "quasichain rule":

$$\boldsymbol{a}_k \equiv \frac{\partial \boldsymbol{r}}{\partial \theta^k} \equiv \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q^i} \frac{\partial \dot{q}^i}{\partial \omega^k}$$

$$= \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q^i} \frac{\partial(\delta q^i)}{\partial(\delta\theta^k)}$$

$$= \sum_{i=1}^{n} A_k^i \, \boldsymbol{e}_i \qquad (k=1,\ldots,n) \tag{26a}$$

The $\boldsymbol{a}_k$ are fundamental to all nonholonomic mechanics considerations. Finally, inverting Eq. (26a) with the help of Eq. (21) readily yields

$$\boldsymbol{e}_i = \sum_{k=1}^{n} \boldsymbol{a}_k \frac{\partial(\delta\theta^k)}{\partial(\delta q^i)}$$

$$= \sum_{k=1}^{n} a_i^k \, \boldsymbol{a}_k \qquad (i=1,\ldots,n) \tag{26b}$$

Remark: Contrary to erroneous statements contained in Ref. 7 (pp. 1076-1077), Eq. (24) for $\delta \boldsymbol{r}$ does not have to be defined (or "chosen properly") in an ad hoc or a posteriori fashion to fit the facts, but instead, as our preceding presentation clearly demonstrates, Eq. (24) flows directly, logically, and uniquely out of the universal virtual displacement definition without approximations and/or guesswork. True principles lead, they do not follow.

III. Reaction-Free Equations of Maggi

Next, inserting the representation [Eq. (24)] into the particle forms [Eq. (7) and (8)] of LP successively yields, from Eqs. (7),

$$0 = S \, \mathrm{d}\boldsymbol{R} \cdot \delta \boldsymbol{r}$$

$$= S \, \mathrm{d}\boldsymbol{R} \cdot \left(\sum_{k=m+1}^{n} \boldsymbol{a}_k \, \delta\theta^k \right)$$

$$= S \, \mathrm{d}\boldsymbol{R} \cdot \left[\sum_{k=m+1}^{n} \left(\sum_{i=1}^{n} A_k^i \, \boldsymbol{e}_i \right) \delta\theta^k \right]$$

$$= \sum_{k=m+1}^{n} \left[\sum_{i=1}^{n} \left(S \, \mathrm{d}\boldsymbol{R} \cdot \boldsymbol{e}_i \right) A_k^i \right] \delta\theta^k$$

$$= \sum_{k=m+1}^{n} \left(\sum_{i=1}^{n} A_k^i \, Q_i^{(r)} \right) \delta\theta^k$$

$$= \sum_{k=m+1}^{n} \Theta_k^{(r)} \, \delta\theta^k \tag{27}$$

from which, since the $(n-m)$ $\delta\theta$ are now independent,

$$\Theta_k^{(r)} \equiv \sum_{i=1}^{n} A_k^i Q_i^{(r)} \tag{28}$$

defined as the covariant kth (nonholonomic) component of system reaction along $\delta\theta^k$, equals zero, for $k=m+1,\ldots,n$. Inserting Eq. (16) into the right side of Eq. (28), and since **a** and **A** are inverse matrices [recall Eqs. (21)], yields

$$\Theta_k^{(r)} = \sum_{i=1}^{n} A_k^i\left(\sum_{l=1}^{m} \lambda_l a_i^l\right)$$

$$= \sum_{l=1}^{m}\left(\sum_{i=1}^{n} a_i^l A_k^i\right)\lambda_l$$

$$= \sum_{l=1}^{m} \delta_k^l \lambda_l$$

$$= \lambda_k, \qquad (=0, \text{ for } k=m+1,\ldots,n) \tag{29}$$

i.e.,

$$\Theta_k^{(r)} \equiv \lambda_k$$

The particle form of Eq. (28) is

$$\Theta_k^{(r)} \equiv S\, \mathrm{d}\boldsymbol{R}\cdot\boldsymbol{a}_k \begin{cases} =0, & (k=m+1,\ldots,n) \\ \neq 0, & (k=1,\ldots,m) \end{cases} \tag{30}$$

since these latter nonzero reactions result from the constraints $\delta\theta_1=\cdots=\delta\theta_m=0$. One can also define $\Theta_{n+1}^{(r)} \equiv S\,\mathrm{d}\boldsymbol{R}\cdot\boldsymbol{a}_{n+1}\neq 0$, required to produce $\delta\theta_{n+1}\equiv\delta t=0$; this will not be needed here, however. From Eq. (8), the insertion of Eq. (24) yields:

$$S\,\mathrm{d}m\boldsymbol{a}\cdot\delta\boldsymbol{r} = S\,\mathrm{d}\boldsymbol{f}\cdot\delta\boldsymbol{r}$$

$$S\,\mathrm{d}m\boldsymbol{a}\cdot\left(\sum_{k=m+1}^{n} \boldsymbol{a}_k\,\delta\theta^k\right) = S\,\mathrm{d}\boldsymbol{f}\cdot\left(\sum_{k=m+1}^{n} \boldsymbol{a}_k\,\delta\theta^k\right)$$

or

$$\sum_{k=m+1}^{n} (I_k-\Theta_k)\delta\theta^k = 0 \tag{31}$$

from which, since the $(n-m)$ $\delta\theta$ are independent,

$$I_k = \Theta_k, \qquad (k=m+1,\ldots,n) \tag{32}$$

Here

$$I_k \equiv S\,\mathrm{d}m\boldsymbol{a}\cdot\boldsymbol{a}_k \tag{33}$$

is the covariant kth (nonholonomic) component of the system *inertia* force, along $\delta\theta^k$, and equals

$$\sum_{i=1}^{n} A_k^i E_i(T)$$

[recall Eqs. (11) and (26)],

$$\Theta_k \equiv S\,\mathrm{d}\boldsymbol{f}\cdot\boldsymbol{a}_k \tag{34}$$

is the covariant kth (nonholonomic) component of the system *impressed/applied* force, along $\delta\theta^k$,

$$\sum_{i=1}^{n} A_k^i Q_i$$

[recall Eqs. (12) and (26)]. These transformations [Eqs. (33) and (34)] hold also for $k=1,\ldots,m$ (and even for $k=n+1$).

Inserting Eqs. (23) into Eqs. (9) and (10) successively yields, for Eq. (9),

$$0 = \sum_{i=1}^{n} Q_i^{(r)}\left(\sum_{k=m+1}^{n} A_k^i\,\delta\theta^k\right)$$

$$= \sum_{k=m+1}^{n} \Theta_k^{(r)}\,\delta\theta^k$$

[i.e., Eqs. (28)] and for Eq. (10),

$$0 = \sum_{i=1}^{n}\left[E_i(T)-Q_i\right]\delta q^i$$

$$= \sum_{i=1}^{n}\left[E_i(T)-Q_i\right]\left(\sum_{k=m+1}^{n} A_k^i\,\delta\theta^k\right)$$

$$= \sum_{k=m+1}^{n}\left(I_k-\Theta_k\right)\delta\theta^k$$

[i.e., Eqs. (32)].

The preceding equations clearly show how to get rid of the reactions $Q_i^{(r)}$ in Eqs. (16) and (17) (i.e., the customary starting point of treatments such as in Refs. 1–4). Multiplying both sides of Eq. (17) with A_k^i $(i=1,\ldots,n;\ k=m+1,\ldots,n)$ and summing over i yields successively

$$\sum_{i=1}^{n} A_k^i E_i(T) = \sum_{i=1}^{n} A_k^i Q_i + \sum_{i=1}^{n} A_k^i\left(\sum_{l=1}^{m}\lambda_l a_i^l\right) \tag{35a}$$

or, since the second right-hand side sum is [by Eq. (29)] zero,

$$\sum_{i=1}^{n} A_k^i E_i(T) = \sum_{i=1}^{n} A_k^i Q_i, \quad \text{or} \quad I_k=\Theta_k \tag{35b}$$

where the final expression is taken from Eq. (32). Further

$$\sum_{i=1}^{n} A_k^i\left(S\,\mathrm{d}m\boldsymbol{a}\cdot\boldsymbol{e}_i\right) = \sum_{i=1}^{n} A_k^i\left(S\,\mathrm{d}\boldsymbol{f}\cdot\boldsymbol{e}_i\right) \tag{35c}$$

or, finally, (particle forms)

$$S\,\mathrm{d}m\boldsymbol{a}\cdot\boldsymbol{a}_k = S\,\mathrm{d}\boldsymbol{f}\cdot\boldsymbol{a}_k, \qquad (k=m+1,\ldots,n) \tag{35d}$$

[i.e., Eq. (32)].

The fundamental reaction-free equations (35b) were derived first in 1894 (and published in 1896) and then again in 1901 by the distinguished Italian mechanician G. A. Maggi.[11-13] Although Maggi's equations are of pivotal importance for the entire theory of nonholonomic equations of motion [since they constitute the bridge between the "multiplier/reactions" equations of Routh/Voss, and the purely nonholonomic reaction-free equations of Boltzmann (1902)/Hamel (1903/1904)], regrettably and inexplicably they seem to be unknown to most contemporary Western mechanicians, including the Italian ones. A certain special case of Maggi's equations is of particular practical significance: one solves the m [Eq. (4)] for the first (dependent) $m\dot{q}$, in terms of the last (independent) $(n-m)$ $\dot{q}$, i.e.,

$$\dot{q}^j = \sum_{i=m+1}^{n} b_i^j\dot{q}^i + b_{n+1}^j, \qquad (j=1,\ldots,m) \tag{36}$$

where b_i^j and b_{n+1}^j are functions of $q_1,\ldots,q_n,t$. The virtual counterpart of Eq. (36) is

$$\delta q^j = \sum_{i=m+1}^{n} b_i^j\delta q^i, \qquad (j=1,\ldots,m) \tag{37}$$

Now, Eqs. (36) and (37) can be viewed as the following special case of Eqs. (19) and (23):

$$\omega^k \equiv \dot{q}^k - \sum_{i=m+1}^{n} b_i^k\,\dot{q}^i - b_{n+1}^k = 0, \qquad (k=1,\ldots,m)$$

$$\omega^k \equiv \dot{q}^k, \qquad (k=m+1,\ldots,n) \tag{38}$$

and

$$\delta\theta^k \equiv \delta q^k - \sum_{i=m+1}^{n} b_i^k \, \delta q^i, \qquad (k=1,\dots,m)$$

$$\delta\theta^k \equiv \delta q^k, \qquad (k=m+1,\dots,n) \tag{39}$$

i.e., in this case

$$\mathbf{a} = \left[\begin{array}{c|c} \mathbf{1} & -\mathbf{b} \\ \hline \mathbf{0} & \mathbf{1} \end{array}\right] \begin{array}{l}(m)\\(n-m)\end{array} \tag{40}$$
$$\qquad (m) \quad (n-m)$$

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{1} & \mathbf{b} \\ \hline \mathbf{0} & \mathbf{1} \end{array}\right] \begin{array}{l}(m)\\(n-m)\end{array} \tag{41}$$
$$\qquad (m) \quad (n-m)$$

where $\mathbf{1} = (m \times m)$ unit (diagonal) matrix, $\mathbf{0} = (n-m) \times m$ zero matrix, and

$$\underset{(n-m)}{\mathbf{b}} = [\ b_i^k\ ]m, \qquad (k=1,\dots,m;\ i=m+1,\dots,n) \tag{42}$$

Only the $(n \times n)$ parts of $\mathbf{a}$ and $\mathbf{A}$ are shown here; their $(n+1)$th row and column do not enter the equations of motion. For this particular choice of quasivelocities [Eqs. (41) and (42)], the Maggi equation (35b) reduces to

$$E_k(T) + \sum_{i=1}^{m} b_k^i \, E_i(T) = \tilde{Q}_k \tag{43a}$$

$$\tilde{Q}_k \equiv Q_k + \sum_{i=1}^{m} b_k^i \, Q_i, \qquad (k=m+1,\dots,n) \tag{43b}$$

To the best of our knowledge, Eqs. (43) were first arrived at by a different method by the distinguished French mathematician and mathematical physicist J. Hadamard in 1895/1899 [Ref. 14, pp. 400–404, Eqs. (5–10); and Ref. 15, pp. 50–54, Eqs. (6), (8) and (9)]. Since then they have been routinely appearing in many (almost exclusively non-English-language) famous expositions, as the direct result of the combination of Eq. (37) with Eq. (10), without explicit recourse to the method of quasicoordinates [e.g., Ref. 17, pp. 395–398, Eqs. (8.7.6); Ref. 18, Eqs. (8) and (9); Ref. 16, pp. 407–408, Eqs. (12); pp. 413–415, Eqs. (10)]. These equations were rediscovered in 1973 by Passerello and Huston[6] in an ad hoc fashion. All recent references, such as Refs. 1–4, are actually rediscovering these Hadamard/Maggi equations by isolated, ungeometrical, and tortuous ways.

The reactionless Maggi equations (35b) have a simple geometrical interpretation: the m nonholonomic constraint equations (6) represent the equations of an $(n-m)$-dimensional plane, the "virtual system hyperplane" (VP), through (q,t) in V_{n+1}, and is tangent to V_{n+1} there. The coefficients a_i^k $(k=1,\dots,m)$, properly normalized, are then the direction cosines of the m-independent system vectors $A^k \equiv \{a_i^k\}$ normal to VP, whereas the remaining $(n-m)$-independent system vectors A_k $(k=m+1,\dots,n)$ lie, or define, or span the VP. [It should be pointed out that Maggi's equations (35b and 43), *cannot* detect the difference between genuinely nonholonomic constraints and holonomic ones in differential form; these equations of motion hold unchanged for Eqs. (4) and (6), like constraint cases. Only Boltzmann/Hamel-type equations can "see" this distinction. This fact does not affect our results here.]

Thus, Eqs. (6), (22), and (23), in system vector form, read

$$\delta\theta^k = \boldsymbol{A}^k \cdot \delta\boldsymbol{q} = 0, \qquad (k=1,\dots,m;\ n+1)$$

$$\delta\theta^k = \boldsymbol{A}^k \cdot \delta\boldsymbol{q} \neq 0, \qquad (k=m+1,\dots,n)$$

$$\delta q^i = \boldsymbol{\alpha}^i \cdot \delta\boldsymbol{\theta} \neq 0, \qquad (i=1,\dots,n)$$

$$\boldsymbol{A}^k = \{a_i^k\}, \qquad \boldsymbol{\alpha}^i = \{A_k^i\}, \qquad (i,\ k=1,\dots,n;\ n+1) \tag{44}$$

where the system virtual displacement vector $\delta\boldsymbol{q}$ is

$$\delta\boldsymbol{q} \equiv \sum_{i=1}^{n} \boldsymbol{\alpha}_i \, \delta q^i = \sum_{k=m+1}^{n} \boldsymbol{A}_k \, \delta\theta^k \equiv \delta\boldsymbol{\theta} \tag{45}$$

and $\boldsymbol{\alpha}_i$, $\boldsymbol{A}_k$ are, respectively, the reciprocal or dual bases to $\boldsymbol{\alpha}^i$ and $\boldsymbol{A}^k$; as is well known, the latter are defined by

$$\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}^j = \delta_i^j \tag{46a}$$

$$\boldsymbol{A}_k \cdot \boldsymbol{A}^l = \delta_k^l \tag{46b}$$

One can thus say that the totality of the $(n-m)$-dimensional "elements" [i.e., the vectors $\delta\boldsymbol{q}$ or $\delta\boldsymbol{\theta} = (\delta\theta^{m+1},\dots,\delta\theta^n)$] that the m constraints [Eq. (6)] order or map to every point (q,t) of the configuration space, spans a nonholonomic manifold (the system VP) imbedded in that space at (q,t). The reactions of these constraints are perpendicular to these elements $\delta\boldsymbol{q} \equiv \delta\boldsymbol{\theta}$.

Therefore, multiplying the Routh/Voss equations (17) by A_k^i and summing over i from 1 to n means taking the dot/scalar product of their system vector

$$\boldsymbol{0} = \boldsymbol{E}(T) - \boldsymbol{Q} - \boldsymbol{Q}^{(r)} \equiv \sum_{i=1}^{n} \left[E_i(T) - Q_i - Q_i^{(r)} \right] \boldsymbol{\alpha}^i$$

with the (nonholonomic) system vectors $\boldsymbol{A}_k$ $(k=m+1,\dots,n)$:

$$0 = \sum_{i=1}^{n} \left[E_i(T) - Q_i - Q_i^{(r)} \right] A_k^i = \sum_{i=1}^{n} \left[E_i(T) - Q_i \right] A_k^i$$

because

$$\boldsymbol{Q}^{(r)} \cdot \boldsymbol{A}_k = \left(\sum_{i=1}^{n} Q_i^{(r)} \boldsymbol{\alpha}^i \right) \cdot \left(\sum_{j=1}^{n} A_k^j \, \boldsymbol{\alpha}_j \right) = \sum_{i,j=1}^{n} Q_i^{(r)} A_k^j \left(\boldsymbol{\alpha}^i \cdot \boldsymbol{\alpha}_j \right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_k^j \, \delta_j^i \right) Q_i^{(r)} = \sum_{i=1}^{n} A_k^i \, Q_i^{(r)} \tag{47}$$

equals zero (= projection of $\boldsymbol{Q}^{(r)}$ onto the system VP). In sum,

$$\boldsymbol{Q}^{(r)} \cdot \boldsymbol{A}_k = \Theta_k^{(r)} \begin{cases} = 0, & (k=m+1,\dots,n) \\ \neq 0, & (k=1,\dots,m;\ n+1) \end{cases} \tag{48a}$$

where

$$\boldsymbol{Q}^{(r)} \equiv \sum_{i=1}^{n} Q_i^{(r)} \, \boldsymbol{\alpha}^i = \sum_{i=1}^{n} Q^{(r)i} \, \boldsymbol{\alpha}_i = \sum_{k=1}^{m} \Theta_k^{(r)} \boldsymbol{A}^k = \sum_{k=1}^{m} \Theta^{(r)k} \boldsymbol{A}_k \equiv \boldsymbol{\Theta}^{(r)} \tag{48b}$$

IV. Calculation of Reactions

General Maggi Case

Let us finally find how to compute the reactions $Q_i^{(r)}$ and $\Theta_k^{(r)} = \lambda_k$. From Eq. (48)

$$\Theta_k^{(r)} \begin{cases} \neq 0, & \delta\theta^k = 0, & (k=1,\dots,m) \\ = 0, & \delta\theta^k \neq 0, & (k=m+1,\dots,n) \end{cases} \tag{49}$$

and, therefore, Eq. (28), with Eqs. (16) and (17), yields

$$\Theta_k^{(r)} = \sum_{i=1}^{n} A_k^i \, Q_i^{(r)}$$

$$= \sum_{i=1}^{n} A_k^i \left[E_i(T) - Q_i \right]$$

$$= \sum_{i=1}^{n} A_k^i \, E_i(T) - \sum_{i=1}^{n} A_k^i \, Q_i$$

$$= I_k - \Theta_k$$

$$\neq 0, \qquad (k=1,\ldots,m) \tag{50}$$

$$= 0, \qquad (k=m+1,\ldots,n) \tag{51}$$

Equations (50) and (51) are the most general nonholonomic system equations of motion. As shown in Ref. 19, they also hold for nonlinear nonholonomic constraints [Eq. (18)] with appropriate generalizations. Further transformations on I_k lead to the equations of Boltzmann/Hamel, Appell, et al.

Critical comments: 1) Eqs. (50) can be extended for the case $k=n+1$ (time coordinate); this, however, does not produce another equation of motion but, instead, a "power or rate of working" theorem, and so it will not be pursued any further here. 2) Equations (50) and (51) can also result if one applies the "principle of relaxation of constraints" along with the familiar method of Lagrangian multipliers. Analytically this means combining LP

$$\sum_{k=1}^{n} (I_k - \Theta_k) \, \delta\theta^k = 0, \qquad (\delta\theta\text{: constrained})$$

with the $m(<n)$ constraints

$$1 \cdot \delta\theta^1 = 0, \ldots, \qquad 1 \cdot \delta\theta^m = 0$$

or

$$\sum_{k=1}^{n} \delta_k^l \cdot \delta\theta^k = 0, \qquad (l=1,\ldots,m)$$

Thus, one obtains, with $[-\Theta_k^{(r)}]$ as the m multipliers [from Eq. (27) and the preceding constraints],

$$\sum_{k=1}^{m} (I_k - \Theta_k - \Theta_k^{(r)}) \delta\theta^k + \sum_{k=m+1}^{n} (I_k - \Theta_k) \, \delta\theta^k$$

$$= 0, \qquad (\delta\theta\text{: unconstrained})$$

from which Eqs. (50) and (51) follow (see also Ref. 19).

3) Whether calculating reactions or not, one should remember to enforce the constraints only at the final stage, i.e., after all differentiations, etc., have been carried out and not before; otherwise, one will obtain incorrect equations.

Special Hadamard/Maggi Case

For the special case [Eqs. (36–42)], the general reaction-containing Maggi's equations (50) and (51) reduce to

α) $$Q_i^{(r)} = \sum_{k=1}^{m} a_i^k \, \Theta_k^{(r)}, \qquad \text{[inverting Eq. (28)]}$$

$$= \sum_{k=1}^{m} \delta_i^k \, \Theta_k^{(r)} = \Theta_i^{(r)} \equiv \tilde{Q}_i^{(r)} \equiv \lambda_i$$

or

$$Q_i^{(r)} = E_i(T) - Q_i, \qquad (i=1,\ldots,m) \tag{52a}$$

from which

$$E_i(T) = Q_i + \lambda_i \tag{52b}$$

β) $$Q_i^{(r)} = \sum_{k=1}^{m} a_i^k \, \Theta_k^{(r)}$$

$$= -\sum_{k=1}^{m} b_i^k \, \Theta_k^{(r)} = -\sum_{k=1}^{m} b_i^k \, \tilde{Q}_k^{(r)}$$

or

$$Q_i^{(r)} = -\sum_{k=1}^{m} b_i^k [E_k(T) - Q_k], \qquad (i=m+1,\ldots,n) \tag{53a}$$

from which

$$E_i(T) = Q_i - \sum_{k=1}^{m} b_i^k \, \lambda_k \tag{53b}$$

Inversely

$$\Theta_k^{(r)} = \sum_{i=1}^{n} A_k^i \, Q_i^{(r)}$$

$$= \sum_{i=1}^{n} \delta_k^i \, [E_i(T) - Q_i]$$

$$= E_k(T) - Q_k \equiv \tilde{Q}_k, \qquad (k=1,\ldots,m) \tag{54}$$

and

$$\Theta_k^{(r)} = \sum_{i=1}^{n} A_k^i \, Q_i^{(r)}$$

$$= \sum_{i=1}^{m} b_k^i \, Q_i^{(r)} + \sum_{i=m+1}^{n} \delta_k^i \, Q_i^{(r)}$$

$$= \sum_{i=1}^{m} b_k^i \, [E_i(T) - Q_i] + E_k(T) - Q_k$$

$$= 0 \qquad (k=m+1,\ldots,n)$$

as expected [recall Eqs. (43)].

Remarks: 1) In view of the purely kinematical result

$$E_i(T) = \frac{\partial S_a}{\partial \ddot{q}^i}$$

where $S_a \equiv S\frac{1}{2}\,\mathrm{d}ma^2$ = Appellian function, or "acceleration energy," which holds always, one can replace $E_i(T)$ throughout this paper with $\partial S_a/\partial \ddot{q}^i$, and call the resulting equations "Appellian forms of Maggi's equations."

2) It can be easily shown that the choices $\partial r/\partial\theta^k$, $\partial v/\partial\omega^k$, $\partial a/\partial\dot{\omega}^k$ (*always* equal to each other and to a_k) for a_k in Eq. (35d) yield, respectively, the equations of Maggi, "Kane," and Appell (see Refs. 19 and 20).

V. Application: Calculation of Reactions in the Knife-Edge or Sled Problem

This well-known problem consists of the following: find the equations of motion of a knife whose rigid blade remains perpendicular to the x-y plane. The knife's contact point with that plane, C, has coordinates (x,y), and its mass center G lies a distance $s(\neq 0)$ from C along the knife. Finally, the knife's instantaneous orientation is described by its angle ϕ with respect to the positive x axis. For additional details see Ref. 5, pp. 465–469 and Refs. 19 and 21. Here we choose $q_1 = x$, $q_2 = y$, $q_3 = \phi$, and, due to the problem's nonholonomic and scleronomic constraint $\mathrm{d}y/\mathrm{d}x = \tan\phi$ (see Fig. 1),

$$\omega_1 = (-\sin\phi)\dot{x} + (\cos\phi)\dot{y} + (0)\dot{\phi} \qquad (=0)$$

$$\omega_2 = (\cos\phi)\dot{x} + (\sin\phi)\dot{y} + (0)\phi$$

$$\omega_3 = (0)\dot{x} + (0)\dot{y} + (1)\dot{\phi} \tag{55}$$

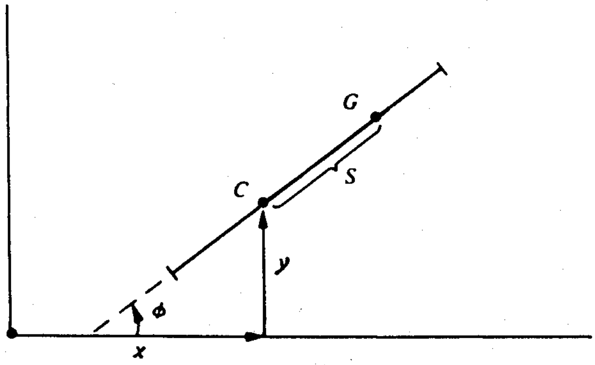

Fig. 1 The Knife edge/sleigh problem.

from which

$$\mathbf{a} = \mathbf{A} = \begin{pmatrix} -\sin\phi & \cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{56}$$

Because of the orthogonality of the coordinates, the difference between covariant and contravariant components and basis vectors disappears, and so only subscripts will be used. The constraint is simply

$$\omega_1 \text{ [= component of velocity of } C \textit{ perpendicular} \text{ to the knife's edge]} \equiv \omega_n = 0 \tag{57}$$

while

$$\omega_2 \text{ [= component of velocity of } C \textit{ along} \text{ the knife's edge]} \equiv \omega_t = v \tag{58}$$

and

$$\omega_3 \equiv \omega_\phi = \dot\phi \tag{59}$$

Clearly here $m = 1$, $n = 3$. Next, from a well-known elementary dynamics formula

$$T = \tfrac{1}{2} m(\dot x^2 + \dot y^2) + \tfrac{1}{2} I\dot\phi^2 + ms\dot\phi(\dot y \cos\phi - \dot x \sin\phi) \tag{60}$$

where m, I = mass and moment of inertia (about an axis through C perpendicular to the x-y plane) of the knife; as explained earlier, the constraint [Eq. (57)] is not to be enforced on Eq. (60). Finally, let $Q_1 \equiv X$, $Q_2 \equiv Y$, and $Q_3 \equiv M$ be the x, y, and z holonomic components of the (given) generalized impressed force (X, Y: forces, M: moment).

With $E_*(T) \equiv E_*$ for convenience, one readily finds

$$E_1 \to E_x = (m\dot x - ms\dot\phi \sin\phi)^\cdot = m\ddot x - ms\dot\phi^2 \cos\phi - ms\ddot\phi \sin\phi \tag{61}$$

$$E_2 \to E_y = (m\dot y + ms\dot\phi \cos\phi)^\cdot = m\ddot y - ms\dot\phi^2 \sin\phi + ms\ddot\phi \cos\phi \tag{62}$$

$$E_3 \to E_\phi = [ms(\dot y \cos\phi - \dot x \sin\phi)]^\cdot + I\ddot\phi + ms\dot\phi(\dot y \sin\phi + \dot x \cos\phi)$$
$$= I\ddot\phi + ms(\ddot y \cos\phi - \ddot x \sin\phi) \tag{63}$$

Applying the previous theory to the preceding problem readily shows that here

$$\alpha_1 = \alpha^1 \equiv \alpha_x = \hat\imath, \quad \alpha_2 = \alpha^2 \equiv \alpha_y = \hat\jmath, \quad \alpha_3 = \alpha^3 \equiv \alpha_\phi = \hat k \tag{64}$$

where $(\hat\imath, \hat\jmath, \hat k)$ are the well-known orthonormal triad along x, y, z,

$$A_1 = A^1 \equiv A_n = (-\sin\phi)\ \alpha_x + (\cos\phi)\ \alpha_y + (0)\ \alpha_\phi \tag{65a}$$

$$A_2 = A^2 \equiv A_t = (\cos\phi)\ \alpha_x + (\sin\phi)\ \alpha_y + (0)\ \alpha_\phi \tag{65b}$$

$$A_3 = A^3 \equiv A_\phi = (0)\ \alpha_x + (0)\ \alpha_y + (1)\ \alpha_\phi \tag{65c}$$

(n, t: along normal and tangential to knife's edge, respectively), from which

$$\alpha_x = (-\sin\phi)\ A_n + (\cos\phi)\ A_t + (0)\ A_\phi \tag{66a}$$

$$\alpha_y = (\cos\phi)\ A_n + (\sin\phi)\ A_t + (0)\ A_\phi \tag{66b}$$

$$\alpha_\phi = (0)\ A_n + (0)\ A_t + (1)\ A_\phi \tag{66c}$$

and, therefore,

$$q = (x, y, \phi) = \text{generalized position vector} \tag{67}$$

$$\begin{aligned} dq/dt &= \text{generalized velocity vector} \\ &= \dot x\ \alpha_x + \dot y\ \alpha_y + \dot\phi\ \alpha_\phi \qquad \text{(in holonomic basis H)} \\ &= (0)A_n + (v)A_t + (\dot\phi)A_\phi \qquad \text{(in nonholonomic basis NH)} \end{aligned} \tag{68}$$

$$\begin{aligned} E(T) \equiv E &= E_x\ \alpha_x + E_y\ \alpha_y + E_\phi\ \alpha_\phi \qquad \text{(H)} \\ &= I_n\ A_n + I_t\ A_t + I_\phi\ A_\phi \qquad \text{(NH)} \end{aligned} \tag{69}$$

$$\begin{aligned} Q &\equiv X\ \alpha_x + Y\ \alpha_y + M\ \alpha_\phi \qquad \text{(H)} \\ &= N\ A_n + K\ A_t + \Phi\ A_\phi \qquad \text{(NH)} \end{aligned} \tag{70}$$

$$\begin{aligned} Q^{(r)} &\equiv X^{(r)}\ \alpha_x + Y^{(r)}\ \alpha_y + M^{(r)}\ \alpha_\phi \qquad \text{(H)} \\ &= N^{(r)}\ A_n + (0)\ A_t + (0)\ A_\phi \qquad \text{(NH)} \end{aligned} \tag{71}$$

Thus the nonholonomic component form of the system equation

$$E = Q + Q^{(r)} \tag{72}$$

is

$$I_n = N + N^{(r)} \tag{73a}$$

$$I_t = K, \qquad (K^{(r)} = 0) \tag{73b}$$

$$I_\phi = \Phi, \qquad (\Phi^{(r)} = 0) \tag{73c}$$

where

$$\begin{aligned} I_n = E \cdot A_n &= A_{xn}\ E_x + A_{yn}\ E_y + A_{\phi n}\ E_\phi \\ &= (-\sin\phi)E_x + (\cos\phi)E_y + (0)E_\phi \\ &= -m\ddot x \sin\phi + m\ddot y \cos\phi + ms\ddot\phi \end{aligned} \tag{74}$$

$$\begin{aligned} I_t = E \cdot A_t &= A_{xt}\ E_x + A_{yt}\ E_y + A_{\phi t}\ E_\phi \\ &= (\cos\phi)E_x + (\sin\phi)E_y + (0)E_\phi \\ &= m\ddot x \cos\phi + m\ddot y \sin\phi - ms\dot\phi^2 \end{aligned} \tag{75}$$

$$I_\phi = \boldsymbol{E}\cdot\boldsymbol{A}_\phi = A_{x\phi}E_x + A_{y\phi}E_y + A_{\phi\phi}E_\phi$$

$$= (0)E_x + (0)E_y + (1)E_\phi = ms(-\ddot{x}\sin\phi + \ddot{y}\cos\phi) + I\ddot{\phi} \tag{76}$$

$$N = \boldsymbol{Q}\cdot\boldsymbol{A}_n = (-\sin\phi)X + (\cos\phi)Y + (0)M \tag{77}$$

$$N^{(r)} = \boldsymbol{Q}^{(r)}\cdot\boldsymbol{A}_n \tag{78}$$

$$K = \boldsymbol{Q}\cdot\boldsymbol{A}_t = (\cos\phi)X + (\sin\phi)Y + (0)M \tag{79}$$

$$[K^{(r)} = \boldsymbol{Q}^{(r)}\cdot\boldsymbol{A}_t = 0] \tag{80a}$$

$$\Phi = \boldsymbol{Q}\cdot\boldsymbol{A}_\phi = (0)X + (0)Y + (1)M \tag{80b}$$

$$[\Phi^{(r)} = \boldsymbol{Q}^{(r)}\cdot\boldsymbol{A}_\phi = 0] \tag{80c}$$

Inserting the results of Eqs. (74–80) to Eqs. (73) finally yields

$$m(\ddot{y}\cos\phi - \ddot{x}\sin\phi) + ms\ddot{\phi}$$

$$= \cos\phi\ Y - \sin\phi X + \cos\phi Y^{(r)} - \sin\phi X^{(r)} \tag{81}$$

$$m(\ddot{x}\cos\phi + \ddot{y}\sin\phi) - ms\dot{\phi}^2 = \cos\phi X + \sin\phi Y \tag{82}$$

$$ms(\ddot{y}\cos\phi - \ddot{x}\sin\phi) + I\ddot{\phi} = M \tag{83}$$

These are the Maggi equations of our problem. Equations (82) and (83) would also result if one eliminated the Lagrangian multiplier from the three Routh/Voss equations of this problem—recall Eqs. (16 and 17). From Eq. (81) that multiplier equals $Y^{(r)}\cdot\cos\phi - X^{(r)}\cdot\sin\phi = \cdots$. Once fully understood, the Maggi procedure can be considerably streamlined. More instructive forms of Eqs. (81–83) result if one uses quasicoordinates: they are the well-known

$$mv\dot{\phi} + ms\ddot{\phi} = N + N^{(r)} \tag{84}$$

$$m\dot{v} - ms\dot{\phi}^2 = K \tag{85}$$

$$msv\dot{\phi} + I\ddot{\phi} = M \tag{86}$$

One first solves the system of Eqs. (85) and (86) (plus appropriate initial conditions) for $v(t)$ and $\phi(t)$, then substitutes them into Eq. (84) and solves for the reaction $N^{(r)}(t)$.

VI. Conclusions

The proper definition of virtual displacement combined with Lagrange's principle led us directly and naturally to the fundamental equations of Maggi, with or without constraint reactions; the geometrical interpretation of these equations was also discussed, along with a special case, the Hadamard/Maggi equations. A simple application to the well-known knife-edge or sled problem was also presented. It is hoped that this exposition, in addition to deriving general equations such as Eqs. (50) and (51), has also helped demonstrate the simplicity and unifying power of the traditional geometrical approach over the currently popular algebraic/matrix one.

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